CIEM5000: Structural Engineering Base

Matrix Method – Final Details

Tom van Woudenberg, Iuri Rocha

The Matrix Method

Main steps:

- Extract element matrices
- Impose nodal equilibrium
- Impose boundary conditions
- Solve for unknown displacements
- Postprocess results

This week:

- Element loads
- Non-zero Dirichlet boundary conditions in two different ways
- Postprocessing: support reactions and element fields
- Matrix method versus FEM — parallels and differences
- Example: Support reactions of extension bar with distributed load
- Example: A fully-resolved example by hand
- Workshop: Wrap up the code and solve a frame structure

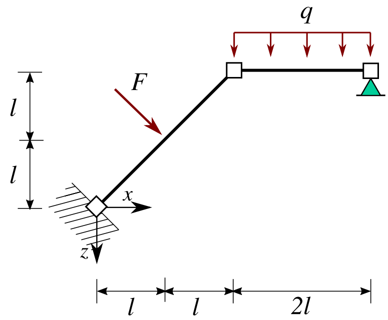
Element loads

The matrix method is a discrete approach

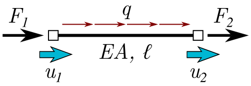
- Nodal loads treated easily
- What if we have loads applied inside elements?

A number of approaches to handle this:

- Further discretization (seldom helps)
- ODE approach
- Work-based approach



Element loads — ODE approach



We can follow the same steps as before:

- General solution for the ODE:

$$EA\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = -q \quad \Rightarrow \quad u(x) = -\frac{qx^2}{2EA} + C_1x + C_2$$

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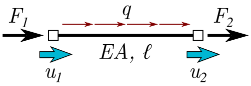

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- Boundary conditions and final solution:

$$u(0) = u_1 \quad u(\ell) = u_2 \quad \Rightarrow \quad C_1 = \frac{q\ell}{2EA} + \frac{u_2 - u_1}{\ell} \quad C_2 = u_1$$

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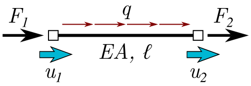

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- Relate forces at the edges with internal stresses:

$$N(x) = \frac{EA}{\ell}(u_2 - u_1) + \frac{q\ell - 2qx}{2} \qquad F_1 = -N_1 \quad F_2 = N_2$$

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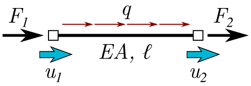

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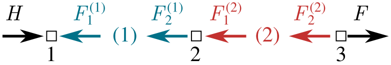
- Relate forces and displacements, but now an extra term appears:

$$\frac{EA}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} \frac{q\ell}{2} \\ \frac{q\ell}{2} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

Dealing with equivalent loads



The new term is an equivalent nodal load:

- Element loads $\Rightarrow$ nodal loads
- Force equilibrium at the nodes therefore changes a bit:

$$-\sum_e \mathbf{f}^e + \mathbf{f}_{\text{nodal}} = \mathbf{0}$$

$$-\sum_e \left(\mathbf{K}^e \mathbf{u}^e - \mathbf{f}_{\text{eq}}^e\right) + \mathbf{f}_{\text{nodal}} = \mathbf{0}$$

$$\sum_e \left(\mathbf{K}^e \mathbf{u}^e\right) = \mathbf{f}_{\text{nodal}} + \sum_e \mathbf{f}_{\text{eq}}^e$$

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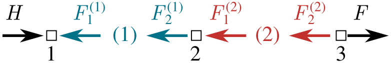

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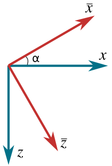
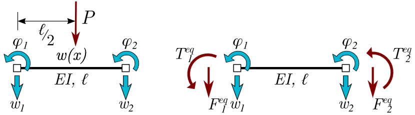
Remember, this is a force in the global coordinate system!

$$\mathbf{f}_{\text{eq}} = \mathbf{T}^{\text{T}}\bar{\mathbf{f}}_{\text{eq}}$$



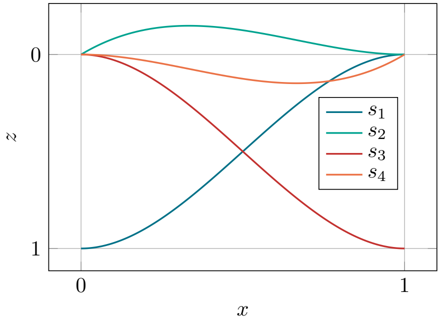
Work-based element loads



Euler-Bernoulli bending, point load at midspan:

- Displacement field for arbitrary DOFs (ODE with $q = 0$):

$$w(x) = \underbrace{\left(\frac{2x^3}{\ell^3} - \frac{3x^2}{\ell^2} + 1\right)}_{s_1} w_1 + \underbrace{\left(-\frac{x^3}{\ell^2} + \frac{2x^2}{\ell} - x\right)}_{s_2} \varphi_1 + \underbrace{\left(-\frac{2x^3}{\ell^3} + \frac{3x^2}{\ell^2}\right)}_{s_3} w_2 + \underbrace{\left(-\frac{x^3}{\ell^2} + \frac{x^2}{\ell}\right)}_{s_4} \varphi_2$$



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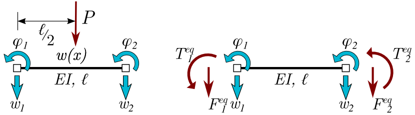

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- Work performed by edge forces:

$$W_F = F_1^{\text{eq}} w_1 + T_1^{\text{eq}} \varphi_1 + F_2^{\text{eq}} w_2 + T_2^{\text{eq}} \varphi_2$$

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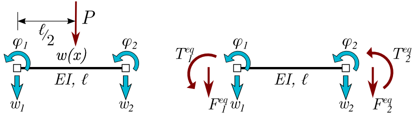

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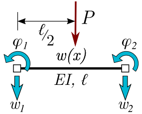

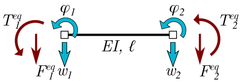

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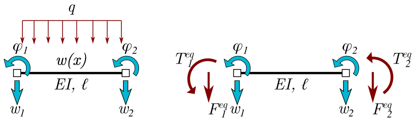

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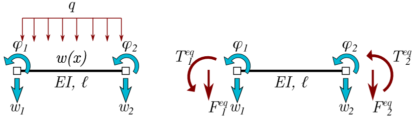

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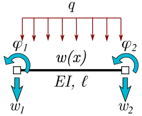

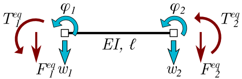

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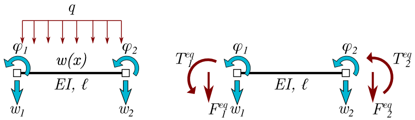

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$$\mathbf{f}_{\mathrm{c}} = \mathbf{K}_{\mathrm{cf}}\mathbf{u}_{\mathrm{f}} + \mathbf{K}_{\mathrm{cc}}\mathbf{u}_{\mathrm{c}}$$

- Note: $\mathbf{f}_{\mathrm{c}}$ includes both nodal loads, equivalent loads and support reactions.

Dirichlet BCs — Size-preserving approach

The approach from before can be annoying to code:

- Reordering the system costs computation time
- Gains when inverting the stiffness matrix are very limited ($N_c \ll N_f$)

Alternatively, we can modify the relevant equations and solve the full system:

- Support reactions recovered later from the unconstrained system

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

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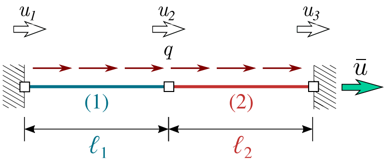
$$\begin{bmatrix} K_{11} & 0 & K_{13} \\ 0 & 1 & 0 \\ K_{31} & 0 & K_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 - K_{12}\Delta_2 \\ \Delta_2 \\ f_3 - K_{32}\Delta_2 \end{bmatrix}$$

Example — element loads and Dirichlet BCs

Let us use what we have just learned and show a quick example:

- Same two-element bar model as before $\Rightarrow$ stiffness matrix does not change!
- Added distributed load and non-zero prescribed displacement on the right



$$\mathbf{K} = \begin{bmatrix} \frac{EA_1}{\ell_1} & -\frac{EA_1}{\ell_1} & 0 \\ -\frac{EA_1}{\ell_1} & \frac{EA_1}{\ell_1} + \frac{EA_2}{\ell_2} & -\frac{EA_2}{\ell_2} \\ 0 & -\frac{EA_2}{\ell_2} & \frac{EA_2}{\ell_2} \end{bmatrix}$$

Element-level postprocessing

From discrete nodal displacements to continuum element fields:

- Assemble and solve the global system of equations:

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- From the ODE solution, recover relevant fields as function of $\overline{\mathbf{u}}^e$:

$$w(x) = \underbrace{\left(\frac{2x^3}{\ell^3} - \frac{3x^2}{\ell^2} + 1\right)}_{s_1} w_1 + \underbrace{\left(-\frac{x^3}{\ell^2} + \frac{2x^2}{\ell} - x\right)}_{s_2} \varphi_1 + \underbrace{\left(-\frac{2x^3}{\ell^3} + \frac{3x^2}{\ell^2}\right)}_{s_3} w_2 + \underbrace{\left(-\frac{x^3}{\ell^2} + \frac{x^2}{\ell}\right)}_{s_4} \varphi_2$$

- Finally, plot the results! Works for displacements and any internal field (e.g. moments)

Matrix method versus FEM

The two methods give the same results for bars. However:

- The matrix method solves the strong form ODEs exactly
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Then why don't we just use the matrix method for everything?

- Gluing elements through equilibrium only works in 1D
- Exact ODE solutions in 2D generally do not exist

Example — 3D frame with torsion

Full solution by hand to demonstrate all steps:

- Definition of a new element (torsion)
- Element reduction for tractability (bending)
- Element loads, support reactions, postprocessing

Values for numerical calculation:

- $EI = 1000$ kNm^2
- $GI_t = 800$ kNm^2
- $\ell = 2$ m
- $T = 4$ kNm
- $q = 6$ kN/m
- $m = 2$ kNm/m