CIEM5000: Structural Engineering Base

Matrix Method — Final Details

Tom van Woudenberg, Iuri Rocha

The Matrix Method

Main steps:

- Extract element matrices
- Impose nodal equilibrium
- Impose boundary conditions
- Solve for unknown displacements
- Postprocess results

This week:

- Element loads
- Non-zero Dirichlet boundary conditions in two different ways
- Postprocessing: support reactions and element fields
- Matrix method versus FEM — parallels and differences
- Example: A fully-resolved example by hand
- Workshop: Wrap up the code and solve a frame structure

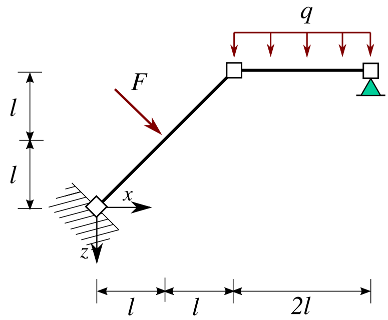
Element loads

The matrix method is a discrete approach

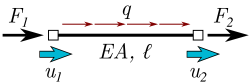
- Nodal loads treated easily
- What if we have loads applied inside elements?

A number of approaches to handle this:

- Further discretization (seldom helps)
- ODE approach
- Work-based approach



Element loads — ODE approach



We can follow the same steps as before:

- General solution for the ODE:

$$EA\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = -q \quad \Rightarrow \quad u(x) = -\frac{qx^2}{2EA} + C_1x + C_2$$

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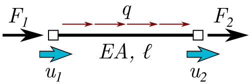

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- Boundary conditions and final solution:

$$u(0) = u_1 \quad u(\ell) = u_2 \quad \Rightarrow \quad C_1 = \frac{q\ell}{2EA} + \frac{u_2 - u_1}{\ell} \quad C_2 = u_1$$

$$u(x) = \frac{q}{2EA}\left(\ell x - x^2\right) + u_1\left(1 - \frac{x}{\ell}\right) + \frac{u_2 x}{\ell}$$

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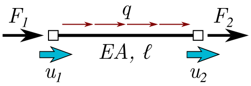

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- Relate forces at the edges with internal stresses:

$$N(x) = \frac{EA}{\ell}\left(u_2 - u_1\right) + \frac{q\ell - 2qx}{2} \qquad F_1 = -N_1 \quad F_2 = N_2$$

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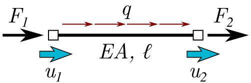

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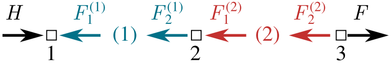
- Relate forces and displacements, but now an extra term appears:

$$\frac{EA}{\ell}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} - \begin{bmatrix}\frac{q\ell}{2}\\ \frac{q\ell}{2}\end{bmatrix} = \begin{bmatrix}F_1\\ F_2\end{bmatrix}$$

Dealing with equivalent loads



The new term is an equivalent nodal load:

- Element loads $\Rightarrow$ nodal loads
- Force equilibrium at the nodes therefore changes a bit:

$$-\sum_e \mathbf{f}^e + \mathbf{f}_{\text{nodal}} = \mathbf{0}$$

$$-\sum_e \left(\mathbf{K}^e \mathbf{u}^e - \mathbf{f}_{\text{eq}}^e\right) + \mathbf{f}_{\text{nodal}} = \mathbf{0}$$

$$\sum_e \left(\mathbf{K}^e \mathbf{u}^e\right) = \mathbf{f}_{\text{nodal}} + \sum_e \mathbf{f}_{\text{eq}}^e$$

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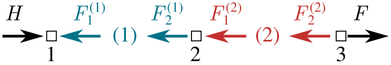

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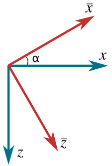
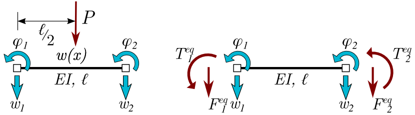
$$\sum_e \left(\mathbf{K}^e\mathbf{u}^e\right) = \mathbf{f}_{\text{nodal}} + \sum_e \mathbf{f}^e_{\text{eq}}$$

Remember, this is a force in the global coordinate system!

$$\mathbf{f}_{\text{eq}} = \mathbf{T}^{\mathrm{T}}\bar{\mathbf{f}}_{\text{eq}}$$



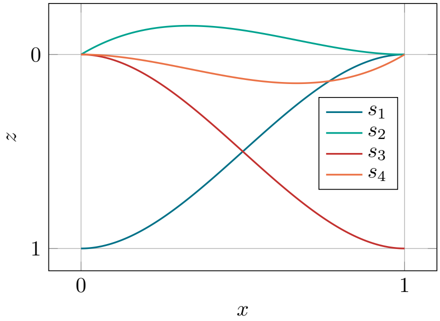
Work-based element loads



Euler-Bernoulli bending, point load at midspan:

- Displacement field for arbitrary DOFs (ODE with $q = 0$):

$$w(x) = \underbrace{\left(\frac{2x^3}{\ell^3} - \frac{3x^2}{\ell^2} + 1\right)}_{s_1} w_1 + \underbrace{\left(-\frac{x^3}{\ell^2} + \frac{2x^2}{\ell} - x\right)}_{s_2} \varphi_1 + \underbrace{\left(-\frac{2x^3}{\ell^3} + \frac{3x^2}{\ell^2}\right)}_{s_3} w_2 + \underbrace{\left(-\frac{x^3}{\ell^2} + \frac{x^2}{\ell}\right)}_{s_4} \varphi_2$$



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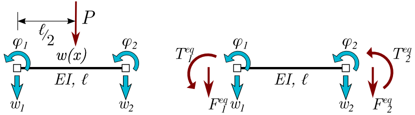

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- Work performed by edge forces:

$$W_F = F_1^{\mathrm{eq}} w_1 + T_1^{\mathrm{eq}} \varphi_1 + F_2^{\mathrm{eq}} w_2 + T_2^{\mathrm{eq}} \varphi_2$$

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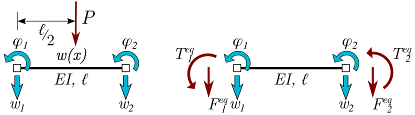

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- Work performed by P under the same displacement:

$$W_q = Pw(\ell/2) = \left(Ps_1(\ell/2)\right) w_1 + \left(Ps_2(\ell/2)\right) \varphi_1 + \left(Ps_3(\ell/2)\right) w_2 + \left(Ps_4(\ell/2)\right) \varphi_2$$

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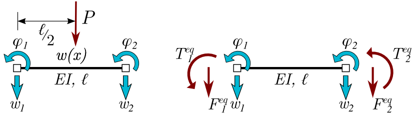

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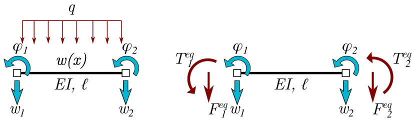

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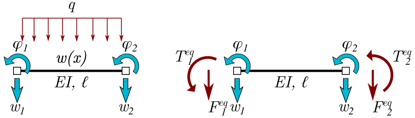

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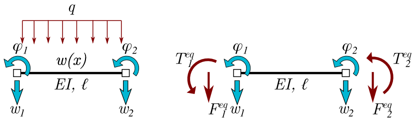

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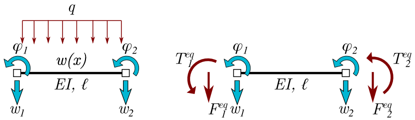

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Up until now displacement BCs have been simple:

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- Note: $\mathbf{f}_{\mathrm{c}}$ includes both nodal loads, equivalent loads and support reactions.

Dirichlet BCs — Size-preserving approach

The approach from before can be annoying to code:

- Reordering the system costs computation time
- Gains when inverting the stiffness matrix are very limited $(N_c \ll N_f)$

Alternatively, we can modify the relevant equations and solve the full system:

- Support reactions recovered later from the unconstrained system

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

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Element-level postprocessing

From discrete nodal displacements to continuum element fields:

- Assemble and solve the global system of equations:

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$$\overline{\mathbf{u}}^e = \mathbf{T}\mathbf{u}^e$$

- From the ODE solution, recover relevant equations as function of $\overline{\mathbf{u}}^e$, e.g.:

$$w(x) = \underbrace{\left(\frac{2x^3}{\ell^3} - \frac{3x^2}{\ell^2} + 1\right)}_{s_1} w_1 + \underbrace{\left(-\frac{x^3}{\ell^2} + \frac{2x^2}{\ell} - x\right)}_{s_2} \varphi_1 + \underbrace{\left(-\frac{2x^3}{\ell^3} + \frac{3x^2}{\ell^2}\right)}_{s_3} w_2 + \underbrace{\left(-\frac{x^3}{\ell^2} + \frac{x^2}{\ell}\right)}_{s_4} \varphi_2$$

- Finally, plot the results! Works for displacements and any other internal field (e.g. moments)

Matrix method versus FEM

The two methods give the same results for bars. However:

- The matrix method solves the strong form ODEs exactly
- FEM solves the weak form problem on the shape function space
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Then why don't we just use the matrix method for everything?

- Gluing elements through equilibrium only works in 1D
- Exact ODE solutions in 2D generally do not exist

Example — 3D frame with torsion

Full solution by hand to demonstrate all steps:

- Definition of a new element (torsion)
- Element reduction for tractability (bending)
- Element loads, support reactions (including distributed loads), postprocessing

Values for numerical calculation:

- $EI = 1000$ kNm^2
- $GI_t = 800$ kNm^2
- $\ell = 2$ m
- $T = 4$ kNm
- $q = 6$ kN/m
- $m = 2$ kNm/m